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Duality and Superconvergence Relation in Supersymmetric Gauge Theories

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Abstract

We investigate the phase structures of various $N = 1$ supersymmetric gauge theories including even the exceptional gauge group from the viewpoint of superconvergence of the gauge field propagator. Especially we analyze in detail whether a new type of duality recently discovered by Oehme in $SU(N_c)$ gauge theory coupled to fundamental matter fields can be found in more general gauge theories with more general matter representations or not. The result is that in the cases of theories including matter fields in *only* the fundamental representation, Oehme's duality holds but otherwise it does not. In the former case, superconvergence relation might give good criterion to describe the interacting non-Abelian Coulomb phase without using some information from dual magnetic theory.

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1 Introduction

Quark confinement is one of the most mysterious properties in quantum field theory. In spite of its obvious existence in experiment, the mechanism of confinement has not been elucidated yet. Of course there have ever been many challenges to understand this phenomena. Lattice formulation of QCD, originally proposed by Wilson [1], is one of them. On the other hand, chiral symmetry breaking is another problem which must be solved. In non-supersymmetric gauge theory, it is considered that these two phenomena, confinement and chiral symmetry breaking, are deeply connected and occur simultaneously in a certain QCD parameter region (strong gauge coupling or small number of quark flavors etc.).

Our main interest here is how the phase structure of QCD changes as the number of quark flavors increases. Naively we expect the following picture: when the number is small, theory is in confinement phase due to its asymptotic freedom property. As the number of flavors is increasing, quarks are deconfined and chiral symmetry is restored before theory becomes asymptotically non-free.

In the intermediate region corresponding to no confinement but asymptotic freedom, we can realize the scale invariant theory because it is expected there exists a non-trivial IR fixed point in this region. The critical value of N_f in $SU(3)$ QCD where quarks are deconfined and chiral symmetry is restored has been evaluated by many authors. Banks and Zaks first pointed out the existence of such a fixed point [2]. They evaluated the first two coefficients of perturbatively expanded β -function which are both gauge and renormalization scheme independent [3]. They showed $N_f^{crit} = 8.05$. Also $N_f^{crit} = 7$ has been obtained in lattice QCD calculation [4]. On the other

hand, Oehme and Zimmermann expected $N_f^{crit} = 10$ using superconvergence relation they called [5]. In this relation, the anomalous dimension of the gauge field as well as the β -function plays an important role. In this way, all the values of N_f^{crit} in different approaches do not coincide each other.

Recently Oehme has applied his superconvergence argument to $N = 1$ supersymmetric $SU(N_c)$ gauge theory and compared with already known results [6]. The phase structure of this theory has been already investigated in detail with the help of so-called “electric-magnetic” duality and holomorphy by Seiberg et al. [7]. Especially Seiberg has insisted on the existence of an interval corresponding to interacting non-Abelian Coulomb phase or conformal window where the theory becomes scale invariant. Consequently Oehme showed quantitative agreement between his argument and the results from Seiberg’s duality. Moreover he also found the important relationship between original electric SUSY theory and dual magnetic one, which might be interpreted as a new type of duality. Since superconvergence arguments can apply for *both SUSY and non-SUSY* theories, the comparison with exact results by Seiberg et al. is very significant.

In this paper, we apply his method to various supersymmetric gauge theories with other gauge groups and other matter contents and check whether the relation he found holds or not in those cases. In section 2, we review the concept of superconvergence of the gauge field propagator in non-supersymmetric case. In section 3, we extend the method in the previous section to the supersymmetric cases and check the Oehme’s duality. Section 4 is devoted to summary and discussions. In Appendices, some basic equations are followed.

2 Superconvergence Relation

Here we shall consider the asymptotic behavior of the gluon propagator at large momentum with the help of the renormalization group (RG) analysis.

First of all, following Oehme and Zimmermann [5], we introduce the operator such as

$$A_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a, \quad (1)$$

where A_μ^a ($\mu = 1, \dots, 4$, and $a = 1, \dots, N_c^2 - 1$) is the $SU(N_c)$ gauge field whose two point function is generally given by

$$\begin{aligned} & \langle 0 | T A_\mu^a(x) A_\nu^b(y) | 0 \rangle \\ &= \int d^4k e^{-ip \cdot (x-y)} \frac{\delta^{ab}}{i} \left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) D(k^2) + \alpha \frac{k_\mu k_\nu}{k^4} \right]. \end{aligned} \quad (2)$$

α is the gauge parameter.

Using (1) and (2), we will obtain

$$\begin{aligned} & \langle 0 | T A_{\mu\nu}^a(x) A_{\rho\sigma}^b(y) | 0 \rangle = \\ & \int d^4k e^{-ip \cdot (x-y)} \frac{\delta^{ab}}{i} (k_\mu k_\rho \delta_{\nu\sigma} - k_\mu k_\sigma \delta_{\nu\rho} - k_\nu k_\rho \delta_{\mu\sigma} + k_\nu k_\sigma \delta_{\mu\rho}) D(k^2). \end{aligned} \quad (3)$$

Scalar function $D(k^2)$ is called “transverse gluon structure function”. Below we shall restrict our consideration to the asymptotic behavior of this function $D(k^2)$ at large momentum.

For that purpose, first we give the normalization condition for $D(k^2)$ as follows:

$$k^2 D(k^2) = 1 \quad \text{at} \quad k^2 = \mu^2, \quad (4)$$

where μ is the normalization point. Then we can write the structure function in the form

$$D = D(k^2, g, \mu^2). \quad (5)$$

g is the gauge coupling constant.

Now, in convenience, let us introduce the dimensionless function R defined by

$$R \equiv k^2 D(k^2, g, \mu^2) = R\left(\frac{k^2}{\mu^2}, g\right). \quad (6)$$

Then Callan-Symanzik equation (RG equation) for the R -function in the Landau gauge ($\alpha = 0$) is given as

$$u \frac{\partial R(u, g)}{\partial u} = \beta(g^2) \frac{\partial R(u, g)}{\partial g^2} + \gamma(g^2) R(u, g), \quad (7)$$

where $u \equiv \frac{k^2}{\mu^2}$ and $\beta(g^2)$ and $\gamma(g^2)$ are the β -function and the anomalous dimension of the gluon field, respectively. In the region of small gauge coupling constant (i.e., at large momentum), they are of the form:

$$\begin{aligned} \beta(g^2) &= g^4(\beta_0 + \beta_1 g^2 + \cdots), \\ \gamma(g^2) &= g^2(\gamma_0 + \gamma_1 g^2 \cdots), \end{aligned} \quad (8)$$

where

$$\begin{aligned} \beta_0 &= -\frac{1}{16\pi^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right), \\ \gamma_0 &= -\frac{1}{16\pi^2} \left(\frac{11}{6} N_c - \frac{4}{3} N_f \right). \end{aligned} \quad (9)$$

N_f denotes the number of quark flavors.

Oehme and Zimmermann solved the eq.(7) in the following form (See Appendix A):

$$R(\frac{k^2}{\mu^2}, g) = R(1, Q) \exp[\int_{g^2}^{Q^2} dx \gamma(x) \beta^{-1}(x)]. \quad (10)$$

Here the effective RG-invariant coupling constant $Q(\frac{k^2}{\mu^2}, g)$ is defined through the equation

$$u \frac{\partial Q^2(u, g)}{\partial u} = \beta(g^2) \frac{\partial Q^2(u, g)}{\partial g^2}. \quad (11)$$

For large momentum, we can show that

$$Q^2(\frac{k^2}{\mu^2}, g) \approx -\frac{1}{\beta_0 \ln \frac{k^2}{\mu^2}}. \quad (12)$$

β_0 is the first coefficient of the β -function defined in eq.(9).

Now we would like to obtain the asymptotic behavior of the R -function given by eq.(10). By substituting (8) and (9) into (10), we can get the following result (see Appendix B)

$$\begin{aligned} R(\frac{k^2}{\mu^2}, g) &= (\frac{Q^2}{g^2})^{\frac{\gamma_0}{\beta_0}} \exp[\int_{g^2}^{Q^2} dx \tau(x)], \\ &\approx C_V (\ln \frac{k^2}{\mu^2})^{-\frac{\gamma_0}{\beta_0}} \end{aligned} \quad (13)$$

where

$$C_V = (g^2 |\beta_0|)^{-\frac{\gamma_0}{\beta_0}} \exp[\int_{g^2}^{Q^2} dx \tau(x)]. \quad (14)$$

Here the function $\tau(x)$ is the regular part of $\frac{\gamma(x)}{\beta(x)}$ at $x = 0$.

Thus leading asymptotic behavior of the R -function at large momentum was determined. Similarly, for the D -function, we find

$$D_{asympt}(k^2) \approx C_V k^{-2} (\ln \frac{k^2}{\mu^2})^{-\frac{\gamma_0}{\beta_0}}. \quad (15)$$

We can conclude here that the asymptotic behavior of the transverse gluon propagator drastically changes due to the sign of γ_0 (or equivalently the sign of $\frac{\gamma_0}{\beta_0}$). If $\gamma_0 < 0$, it converges. While if $\gamma_0 > 0$, it diverges where we have assumed the asymptotic freedom of the theory which means $\beta_0 < 0$.

Let us consider such a case that β_0 and γ_0 are both negative, i.e., the ratio $\frac{\gamma_0}{\beta_0}$ is positive. For the scalar part of the transverse gluon propagator $D(k^2)$, we can apply the Lehmann representation as

$$D(k^2) = \int_0^\infty dm^2 \frac{\rho(m^2)}{m^2 - k^2}, \quad (16)$$

where $\rho(k^2)$ is called the spectral function and is given as the absorptive part of the D -function:

$$\pi\rho(k^2) = \text{Im}D(k^2) = k^{-2}\text{Im}R(k^2). \quad (17)$$

Therefore, for the limit $k^2 \rightarrow -\infty$, we find

$$\rho_{asympt} \approx -\frac{\gamma_0}{\beta_0} C_V k^{-2} \left(\ln \frac{k^2}{\mu^2}\right)^{-\frac{\gamma_0}{\beta_0}-1}. \quad (18)$$

Combining with (15), (16) and (18), a kind of sum rule, called superconvergence relation can be obtained (see Appendix C)

$$\int_0^\infty dm^2 \rho(m^2) = 0 \quad \text{for} \quad \frac{\gamma_0}{\beta_0} > 0. \quad (19)$$

We can show that superconvergence relation obtained here gives some circumstantial evidence for color confinement. In [8], it has been said that superconvergence relation connects with various interpretations of color confinement, such as metric cancellation, the bag model picture and the area law behavior of the Wilson loop in lattice QCD.

On the contrary, in the region where superconvergence relation does not hold (i.e., $\beta_0 < 0$ and $\gamma_0 > 0$), quarks are deconfined and chiral symmetry restored. Moreover it is expected that there exist a non-trivial IR fixed point in this region and the gauge coupling cannot be too strong to confine quarks and to occur quark condensate. In the following section, we will restrict our considerations to this interval in supersymmetric gauge theories.

3 Duality and Superconvergence Relation in various $N = 1$ SUSY Gauge Theories

In the previous section, we discussed superconvergence relation (a kind of sum rule for the spectral function of the transverse gluon propagator) and commented its relation with color confinement. There the essential point was that there existed the region in which we have asymptotic freedom of the theory but no confinement. In such a region, theory may have a non-trivial infrared fixed point at non-vanishing value of the gauge coupling.

On the other hand, Oehme has recently applied the method of superconvergence relation to $N=1$ supersymmetric gauge theory ($SU(N_c)$ gauge theory with fundamental chiral supermultiplet of N_f flavors) [6] and compared the result with the one Seiberg et al. have already obtained using holomorphy and “electric-magnetic” duality [7]. Oehme has insisted in [6] that his result is in quantitative agreement with Seiberg’s duality argument. Moreover he showed that there was an interesting relationship between the coefficients β_0 and γ_0 (defined in the previous section) of the original (electric) theory and those of the dual (magnetic) theory. This may be considered as a new type of duality.

Thus in this section, we first review the analysis by Oehme in [6] and

then try to apply his method to other models of $N = 1$ SUSY gauge theories, which have different gauge groups and different matter representations, to check Oehme's duality. That is the main purpose of this paper.

(I) Oehme's analysis in $N=1$ $SU(N_c)$ gauge theory with massless matter fields in the fundamental representation of N_f flavors

In this case, the one loop coefficients β_0 and γ_0 are given by

$$\begin{aligned}\beta_0 &= -\frac{1}{16\pi^2}(3N_c - N_f), \\ \gamma_0 &= -\frac{1}{16\pi^2}\left(\frac{3}{2}N_c - N_f\right).\end{aligned}\tag{20}$$

If we demand $\beta_0 < 0$ (asymptotic freedom) and $\gamma_0 > 0$ (deconfinement), then we will have the following result:

$$\frac{3}{2}N_c < N_f < 3N_c.\tag{21}$$

This interval is just corresponding to the one Seiberg called interacting non-Abelian Coulomb phase or conformal window. The theory in this interval has a non-trivial infrared fixed point and becomes scale invariant.

On the other hand, as has been well known, we have the dual magnetic description for the original electric theory in the region (21). The gauge group of dual theory is $G_{dual} = SU(N_f - N_c)$ with $N_f^{dual} = N_f$ flavors of magnetic chiral superfields and a certain number of singlet massless superfields. In this dual theory, one-loop coefficients β_0^d and γ_0^d are

$$\begin{aligned}\beta_0^d &= -\frac{1}{16\pi^2}(2N_f - 3N_c), \\ \gamma_0^d &= -\frac{1}{16\pi^2}\left(\frac{1}{2}N_f - \frac{3}{2}N_c\right).\end{aligned}\tag{22}$$

From eqs.(20) and (22), we can extract the relation such as

$$\beta_0^d(N_f) = -2\gamma_0(N_f),$$

$$\beta_0(N_f) = -2\gamma_0^d(N_f). \quad (23)$$

This may be viewed as a new type of duality Oehme first discovered in [6]! Note here that the variable N_f on both sides refers to matter fields with different quantum numbers, i.e., one is electric, the other is magnetic. Eq.(23) might be also interpreted as follows: originally in Seiberg's duality argument the interval (21) has been determined from the requirement of asymptotic freedom in both electric and magnetic theories, which means $\beta_0 < 0$ and $\beta_0^d < 0$. We believe here, however, that we could find a set of parameters describing the interval (21) *only* from those of the original electric theory. Then anomalous dimension γ_0 might be a candidate. Below we shall give the same considerations to various gauge theories with various matter representations to check whether a new type of duality (23) proposed by Oehme is satisfied.

(II) $G = SO(N_c)$ with fundamental matters of N_f flavors

Next we shall investigate the case of $SO(N_c)$ gauge theory with massless superfields of N_f flavors in the fundamental representation. In this case β_0 and γ_0 are given by [9]

$$\begin{aligned} \beta_0 &= -\frac{1}{16\pi^2}[3(N_c - 2) - N_f], \\ \gamma_0 &= -\frac{1}{16\pi^2}[\frac{3}{2}(N_c - 2) - N_f]. \end{aligned} \quad (24)$$

Requiring $\beta_0 < 0$ and $\gamma_0 > 0$, we obtain

$$\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2). \quad (25)$$

This is nothing but the region corresponding to the conformal window discussed in [10]. Then corresponding dual magnetic theory is $G_{dual} = SO(N_f -$

$N_c + 4$) gauge theory with fundamental matter fields of N_f flavors. Dual one-loop coefficients become as

$$\begin{aligned}\beta_0^d &= -\frac{1}{16\pi^2}[2N_f - 3(N_c - 2)], \\ \gamma_0^d &= -\frac{1}{16\pi^2}[\frac{1}{2}N_f - \frac{3}{2}(N_c - 2)].\end{aligned}\tag{26}$$

Thus we can conclude

$$\begin{aligned}\beta_0^d(N_f) &= -2\gamma_0(N_f), \\ \beta_0(N_f) &= -2\gamma_0^d(N_f).\end{aligned}\tag{27}$$

In this case we find Oehme's duality also holds.

(III) $G = Sp(2N_c)$ with fundamental matter fields of N_f flavors

This theory has also the magnetic description in a certain values of N_c and N_f . Dual gauge theory is $G_{dual} = Sp(2N_f - 2N_c - 4)$ with fundamental matter superfields of N_f flavors [11]. The one-loop coefficients of both theories are given as follow:

$$G = Sp(2N_c) \quad \text{“electric”} \quad N = 1 \quad SUSY$$

$$\begin{aligned}\beta_0 &= -\frac{1}{16\pi^2}[3(2N_c + 2) - 2N_f] \\ \gamma_0 &= -\frac{1}{16\pi^2}[\frac{3}{2}(2N_c + 2) - 2N_f] \ ,\end{aligned}\tag{28}$$

and

$$G = Sp(2N_f - 2N_c - 4) \quad \text{“magnetic”} \quad N = 1 \quad SUSY$$

$$\beta_0^d = -\frac{1}{16\pi^2}[4N_f - 3(2N_c + 2)]$$

$$\gamma_0^d = -\frac{1}{16\pi^2}[N_f - \frac{3}{2}(2N_c + 2)] . \quad (29)$$

From (28) and (29), we get the results $\beta_0^d = -2\gamma_0$ and $\beta_0 = -2\gamma_0^d$ in this case, too.

(IV) $G = G_2$ with fundamental matter fields of N_f flavors

This case may be a little non-trivial. Original electric gauge theory has $G = G_2$ with N_f flavors of massless superfields in the fundamental representation. In this case [12],

$$\begin{aligned} \beta_0 &= -\frac{1}{16\pi^2}(12 - N_f), \\ \gamma_0 &= -\frac{1}{16\pi^2}(6 - N_f). \end{aligned} \quad (30)$$

Then the interval $6 < N_f < 12$ corresponds to interacting non-Abelian Coulomb phase as discussed in [12]. There exists the magnetic description in this interval. Dual magnetic gauge theory has $G_{dual} = SU(N_f - 4)$ with fundamental N_f massless matter fields. The one-loop coefficients are given by

$$\begin{aligned} \beta_0^d &= -\frac{1}{16\pi^2}[3(N_f - 4) - N_f] = -\frac{1}{16\pi^2}(2N_f - 12), \\ \gamma_0^d &= -\frac{1}{16\pi^2}[\frac{3}{2}(N_f - 4) - N_f] = -\frac{1}{16\pi^2}(\frac{1}{2}N_f - 6). \end{aligned} \quad (31)$$

Compared (30) with (31), we can come to the conclusion that Oehme's duality is satisfied. This seems to be rather non-trivial check for this duality.

(V) $G = SU(N_c), SO(N_c), Sp(2N_c)$ with massless adjoint matter

After a short time of the discovery of original non-Abelian dualities by Seiberg et al., Kutasov extended Seiberg's argument to $SU(N_c)$ gauge theory including not only fundamental but also adjoint matter superfield [13]. In

this case the dual magnetic theory becomes an $SU(2N_f - N_c)$ gauge theory with N_f massless matter fields in the fundamental representation and an adjoint field and a certain number of singlet massless matter fields. Then the one-loop coefficients in both theories are of the form

$$\begin{aligned}\beta_0 &= -\frac{1}{16\pi^2}(2N_c - N_f), \\ \gamma_0 &= -\frac{1}{16\pi^2}\left(\frac{1}{2}N_c - N_f\right).\end{aligned}\tag{32}$$

and

$$\begin{aligned}\beta_0^d &= -\frac{1}{16\pi^2}(2N_f - 3N_c), \\ \gamma_0^d &= -\frac{1}{16\pi^2}\left(-\frac{1}{2}N_c\right).\end{aligned}\tag{33}$$

We find in this case Oehme's duality checked above does not hold!

Similar analysis can be done for $SO(N_c)$ and $Sp(2N_c)$ gauge groups with adjoint fields [14] and leads us to the same results, i.e., Oehme's duality condition is not also satisfied in these cases.

(VI) $G = SU(N_c)$ with an antisymmetric tensor field

As the final example, we investigate the theory of $G = SU(N_c)$ with an antisymmetric tensor field originally discussed in [15]. Remarkably the dual magnetic gauge group does not become simple but product one in this case. And this theory is also attractive as a model of the supersymmetry breaking [15].

The one-loop coefficients in the original electric theory are

$$\begin{aligned}\beta_0 &= -\frac{1}{16\pi^2}(2N_c - N_f + 3), \\ \gamma_0 &= -\frac{1}{16\pi^2}\left(\frac{1}{2}N_c - N_f + 3\right).\end{aligned}\tag{34}$$

For $N_f > 5$, dual magnetic gauge theory exists. It is represented as the product gauge group $SU(N_f - 3) \otimes Sp(N_f - 4)$ with five species of dual quark superfields: a field transforming as a fundamental under both groups, a conjugate antisymmetric tensor, a fundamental and N_f antifundamentals of $SU(N_f - 3)$ and also $N_c + N_f - 4$ fundamentals of $Sp(N_f - 4)$.

Then one-loop coefficients in each group are given as follows:

$$\begin{aligned}\beta_0^d &= -\frac{1}{16\pi^2}(2N_f - \frac{9}{2}), \\ \gamma_0^d &= -\frac{1}{16\pi^2}(\frac{1}{2}N_f - 3) \quad \text{in } SU(N_f - 3).\end{aligned}\tag{35}$$

and

$$\begin{aligned}\beta_0^d &= -\frac{1}{16\pi^2}(\frac{5}{2}N_f - \frac{1}{2}N_c - \frac{9}{2}), \\ \gamma_0^d &= -\frac{1}{16\pi^2}(N_f - \frac{1}{2}N_c - \frac{3}{2}) \quad \text{in } Sp(N_f - 4).\end{aligned}\tag{36}$$

Thus we conclude from these equations that Oehme's duality relation also does not hold in this final example.

4 Discussions

In this paper, we discussed superconvergence relation following to the original work by Oehme and Zimmermann and then investigated whether a new type of duality recently proposed by Oehme in $N=1$ supersymmetric gauge theory also holds in other gauge theories which have different gauge groups and different matter contents.

As a result, we found that in the cases of gauge groups with matter fields *only* in the fundamental representation, Oehme's duality was satisfied while in those of the theories including adjoint or antisymmetric tensor matter

fields, it was not. The reason may be as follows: when we add an adjoint field to a theory, one can add a superpotential. The model without a superpotential will presumably flow in the infrared to a fixed point, while adding the superpotential drives the system to a new fixed point [13]. Kutasov's duality holds only in the model with a superpotential. On the other hand, in superconvergence argument in which we evaluate β_0 and γ_0 in equation (8), we cannot distinguish a theory with a superpotential from the one without it because the contribution of the superpotential to β -function is higher order one. Actually it appears not in β_0 but in β_1 in eq.(8).

Also we have to consider the problem of the gauge dependence of our method [16]. Originally gluon propagator is unphysical, because it depends on the gauge parameter α . In this paper, we chose Landau gauge ($\alpha = 0$) in convenience. Certainly the value of β_0 does not depend on the specific gauge choice while that of γ_0 does. Therefore even if we obtain the result $\gamma_0 < 0$ in Landau gauge, $\gamma_0 > 0$ might be realized when we move to other gauges. If so, our superconvergence argument based on the value of γ_0 might not be believed. We shall reconsider this point in another occasion.

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Appendices

Appendix A: Proof of equation (10)

In Appendix A, we prove the equation (10). Starting point is

$$u \frac{\partial R(u, g)}{\partial u} = \beta(g^2) \frac{\partial R(u, g)}{\partial g^2} + \gamma(g^2) R(u, g). \quad (37)$$

Here we define \tilde{R} as follows:

$$\tilde{R} \equiv \tilde{R}(Q^2(u, g), g) = R(u, g). \quad (38)$$

Then we find (7) can be rewritten as

$$u \frac{\partial Q^2}{\partial u} \Big|_g \frac{\partial \tilde{R}}{\partial Q^2} \Big|_g = \beta \left(\frac{\partial \tilde{R}}{\partial g^2} \Big|_u + \frac{\partial Q^2}{\partial g^2} \Big|_u \frac{\partial \tilde{R}}{\partial Q^2} \Big|_u \right) + \gamma \tilde{R}. \quad (39)$$

On the other hand, the effective coupling constant Q is through the equation (11), i.e.,

$$u \frac{\partial Q^2}{\partial u} \Big|_g = \beta(g^2) \frac{\partial Q^2}{\partial g^2} \Big|_u. \quad (40)$$

Substituting this relation (40) into (39), we will obtain

$$0 = \beta(g^2) \frac{\partial \tilde{R}(Q^2, g^2)}{\partial g^2} + \gamma(g^2) \tilde{R}(Q^2, g^2), \quad (41)$$

and we can easily solve this equation as the following form

$$\tilde{R}(Q^2, g^2) = \tilde{R}(Q^2, Q^2) \exp \left[\int_{g^2}^{Q^2} dx \frac{\gamma(x)}{\beta(x)} \right]. \quad (42)$$

Moreover, by definition,

$$\begin{aligned} R(1, g^2) &= \tilde{R}(Q^2(1, g^2), g^2), \\ Q^2(1, g^2) &= g^2. \end{aligned} \quad (43)$$

From these equations we obtain $\tilde{R}(Q^2(1, g^2), g^2) = R(1, Q^2)$ and finally we have the result as

$$R(\frac{k^2}{\mu^2}, g^2) = R(1, Q) \exp[\int_{g^2}^{Q^2} dx \frac{\gamma(x)}{\beta(x)}]. \quad Q.E.D. \quad (44)$$

Appendix B: Proof of (13) with (14)

In Appendix B, we evaluate eq.(10) proved in the Appendix A at large momentum and show that (13) is satisfied. For that purpose we first separate the integrand of (10) in the following form

$$\gamma(x)\beta^{-1}(x) = \frac{\gamma_0}{\beta_0 x} + \tau(x), \quad (45)$$

where $\tau(x)$ is corresponding to the regular part of $\gamma(x)\beta^{-1}(x)$ at nearly $x = 0$. Note here that at large momentum, the functions $\beta(x)$ and $\gamma(x)$ are given by (8) respectively in which g^2 is replaced by x .

Then it is not difficult to show (13) and (14). In fact (here we put $R(1, Q) = 1$),

$$\begin{aligned} R(\frac{k^2}{\mu^2}, g^2) &= \exp[\int_{g^2}^{Q^2} dx \frac{\gamma_0}{\beta_0 x}] \exp[\int_{g^2}^{Q^2} dx \tau(x)], \\ &= (\frac{Q^2}{g^2})^{\frac{\gamma_0}{\beta_0}} \exp[\int_{g^2}^{Q^2} dx \tau(x)], \\ &\approx [g^2 |\beta_0| \ln(\frac{k^2}{\mu^2})]^{-\frac{\gamma_0}{\beta_0}} \exp[\int_{g^2}^{Q^2} dx \tau(x)], \\ &\equiv C_V (\ln \frac{k^2}{\mu^2})^{-\frac{\gamma_0}{\beta_0}}, \end{aligned} \quad (46)$$

where

$$C_V = (g^2 |\beta_0|)^{-\frac{\gamma_0}{\beta_0}} \exp[\int_{g^2}^{Q^2} dx \tau(x)]. \quad Q.E.D. \quad (47)$$

Appendix C: Proof of Superconvergence Relation (19)

In this Appendix, we prove the equation (19), the superconvergence relation which is main subject of this paper. To do so, let us use (15), (16) and (18) and combine them.

First multiplying (16) into k^2

$$k^2 D(k^2) = \int_0^\infty dm^2 \frac{k^2 \rho(m^2)}{m^2 - k^2}. \quad (48)$$

Then we can show the following results:

$$\begin{aligned} & \lim_{k^2 \rightarrow -\infty} (l.h.s. of eq.(48)) \\ &= \lim_{k^2 \rightarrow -\infty} k^2 D(k^2) = \lim_{k^2 \rightarrow -\infty} k^2 D_{asympt}(k^2) \\ &= \lim_{k^2 \rightarrow -\infty} C_V (\ln \frac{k^2}{\mu^2})^{-\frac{\gamma_0}{\beta_0}} = 0 \quad \text{for } \frac{\gamma_0}{\beta_0} > 0, \end{aligned} \quad (49)$$

on the other hand,

$$\begin{aligned} & \lim_{k^2 \rightarrow -\infty} (r.h.s. of eq.(48)) \\ &= \lim_{k^2 \rightarrow -\infty} \int_0^\infty dm^2 \frac{k^2 \rho(m^2)}{m^2 - k^2} \\ &= \lim_{k^2 \rightarrow -\infty} \int_0^\infty dm^2 \frac{-(m^2 - k^2)\rho(m^2) + m^2 \rho(m^2)}{m^2 - k^2} \\ &= - \int_0^\infty dm^2 \rho(m^2) + \lim_{k^2 \rightarrow -\infty} \int_0^\infty dm^2 \frac{m^2 \rho(m^2)}{m^2 - k^2}. \end{aligned} \quad (50)$$

Here we compute more in detail the integration in the last column:

$$\begin{aligned} & \lim_{k^2 \rightarrow -\infty} \int_0^\infty dm^2 \frac{m^2 \rho(m^2)}{m^2 - k^2} \\ &= \lim_{k^2 \rightarrow -\infty} \left(\int_0^{\Lambda^2} dm^2 \frac{m^2 \rho(m^2)}{m^2 - k^2} + \int_{\Lambda^2}^\infty dm^2 \frac{m^2 \rho(m^2)}{m^2 - k^2} \right), \\ &= 0 + \lim_{k^2 \rightarrow -\infty} \int_{\Lambda^2}^\infty dm^2 \frac{m^2 \rho_{asympt}(m^2)}{m^2 - k^2}, \\ &= \lim_{k^2 \rightarrow -\infty} \int_{\Lambda^2}^\infty dm^2 \frac{-\frac{\gamma_0}{\beta_0} C_V (\ln \frac{m^2}{\mu^2})^{-\frac{\gamma_0}{\beta_0}-1}}{m^2 - k^2} = 0. \end{aligned} \quad (51)$$

Compared with both sides of (48), we have the result that

$$\int_0^\infty dm^2 \rho(m^2) = 0. \quad Q.E.D. \quad (52)$$

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